POSITIVE DEFINITE FUNCTIONS AND MULTIDIMENSIONAL VERSIONS OF RANDOM VARIABLES

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Abstract. We say that a random vector $X = (X_1, ..., X_n)$ in $\mathbb{R}^n$ is an n-dimensional version of a random variable Y if for any $a \in \mathbb{R}^n$ the random variables $\sum a_i X_i$ and $\gamma(a)Y$ are identically distributed, where $\gamma : \mathbb{R}^n \to [0, \infty)$ is called the standard of X. An old problem is to characterize those functions γ that can appear as the standard of an n-dimensional version. In this paper, we prove the conjecture of Lisitsky that every standard must be the norm of a space that embeds in L_0. This result is almost optimal, as the norm of any finite dimensional subspace of L_p with $p \in (0, 2]$ is the standard of an n-dimensional version (p-stable random vector) by the classical result of P.Lèvy. An equivalent formulation is that if a function of the form $f(\|\cdot\|_K)$ is positive definite on $\mathbb{R}^n$, where K is an origin symmetric star body in $\mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ is an even continuous function, then either the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 or f is a constant function. Combined with known facts about embedding in L_0, this result leads to several generalizations of the solution of Schoenberg's problem on positive definite functions.

1. Introduction

Following Eaton [E], we say that a random vector $X = (X_1, ..., X_n)$ is an *n-dimensional version* of a random variable Y if there exists a function $\gamma : \mathbb{R}^n \to \mathbb{R}$, called the *standard* of X, such that $\gamma(a) > 0$ for every $a \in \mathbb{R}^n,\ a \neq 0$, and for every $a \in \mathbb{R}^n$ the random variables

$$\sum_{i=1}^{n} a_i X_i \quad \text{and} \quad \gamma(a)Y \tag{1}$$

are identically distributed. We assume that $n \geq 2$ and $P\{Y = 0\} < 1$. A problem posed by Eaton is to characterize all n-dimensional versions, and, in particular, characterize all functions γ that can appear as the standard of an n-dimensional version.

It is easily seen [M3], [Ku] that every standard γ is an even homogeneous of degree 1 non-negative (and equal to zero only at zero) continuous function on $\mathbb{R}^n$. This means that $\gamma = \|\cdot\|_K$ is the Minkowski

functional of some origin symmetric star body K in $\mathbb{R}^n$. Recall that a closed bounded set K in $\mathbb{R}^n$ is called *a star body* if every straight line passing through the origin crosses the boundary of K at exactly two points, the origin is an interior point of K and *the Minkowski functional* of K defined by $\|x\|_K = \min\{s \geq 0 : \ x \in sK\}$ is a continuous function on $\mathbb{R}^n$. Note that the class of star bodies includes convex bodies containing the origin in their interior.

Eaton [E] proved that a random vector is an n-dimensional version with the standard $\|\cdot\|_K$ if and only if its characteristic functional has the form $f(\|\cdot\|_K)$, where K is an origin symmetric star body in $\mathbb{R}^n$ and f is an even continuous non-constant function on $\mathbb{R}$ (see also [K3, Lemma 6.1]). By Bochner's theorem, this means that the function $f(\|\cdot\|_K)$ is positive definite. Recall that a complex valued function f defined on $\mathbb{R}^n$ is called *positive definite* on $\mathbb{R}^n$ if, for every finite sequence $\{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ and every choice of complex numbers $\{c_i\}_{i=1}^m$, we have

$$\sum_{i=1}^m \sum_{j=1}^m c_i \bar{c}_j f(x_i - x_j) \geq 0.$$

Thus, Eaton's problem is equivalent to characterizing the classes $\Phi(K)$ consisting of even continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(\|\cdot\|_K)$ is a positive definite function on $\mathbb{R}^n$. In particular, $\|\cdot\|_K$ appears as the standard of an n-dimensional version if and only if the class $\Phi(K)$ is non-trivial, i.e. contains at least one non-constant function. In some places throughout the paper we write $\Phi(E_K)$ instead of $\Phi(K)$, where $E_K = (\mathbb{R}^n, \|\cdot\|_K)$ is the space whose unit ball is K.

The problem of characterization of positive definite norm dependent functions has a long history and goes back to the work of Lèvy and Schoenberg in the 1930s. Lèvy [Le] proved that, for any finite dimensional subspace $(\mathbb{R}^n, \|\cdot\|)$ of L_q with $0 < q \leq 2$, the function $g = \exp(-\|\cdot\|^q)$ is positive definite on $\mathbb{R}^n$, and any random vector $X = (X_1, ..., X_n)$ in $\mathbb{R}^n$, whose characteristic functional is g, satisfies the property (1). This result gave a start to the theory of stable processes that has numerous applications to different areas of mathematics. The concept of an n-dimensional version is a generalization of stable random vectors.

In 1938, Schoenberg [S1,S2] found a connection between positive definite functions and the embedding theory of metric spaces. In particular, Schoenberg [S1] posed the problem of finding the exponents $0 < p \leq 2$ for which the function $\exp(-\|\cdot\|_q^p)$ is positive definite on $\mathbb{R}^n$, where

$$\|x\|_q = (|x_1|^q + ... + |x_n|^q)^{1/q}$$

is the norm the space ℓ_q^n with $2 < q \le \infty$. This problem had been open for more than fifty years. For $q = \infty$, the problem was solved in 1989 by Misiewicz [M2], and for $2 < q < \infty$, the answer was given in [K1] in 1991 (note that, for $1 \le p \le 2$, Schoenberg's question was answered earlier by Dor [D], and the case $n = 2,\ 0 < p \le 1$ was established in [F], [H], [L]). The answers turned out to be the same in both cases: the function $\exp(-\|\cdot\|_q^p)$ is not positive definite for any $p \in (0, 2]$ if $n \ge 3$, and for $n = 2$ the function is positive definite if and only if $0 < p \le 1$. Different and independent proofs of Schoenberg's problems were given by Lisitsky [Li1] and Zastavnyi [Z1, Z2] shortly after the paper [K1] appeared. For generalizations of the solution of Schoenberg's problem, see [KL].

The solution of Schoenberg's problem can be interpreted in terms of isometric embeddings of normed spaces. In fact, the result of Bretagnolle, Dacunha-Castelle and Krivine [BDK] shows that a normed space embeds isometrically in $L_p,\ 0 < p \le 2$ if and only if the function $\exp(-\|\cdot\|^p)$ is positive definite. Hence, the answer to Schoenberg's problem means that that the spaces $\ell_q^n,\ q > 2,\ n \ge 3$ do not embed isometrically in L_p with $0 < p \le 2$.

The classes $\Phi(K)$ have been studied by a number of authors. Schoenberg [S2] proved that $f \in \Phi(\ell_2^n)$ if and only if

$$f(t) = \int_0^\infty \Omega_n(tr)\ d\lambda(r)$$

where $\Omega_n(|\cdot|_2)$ is the Fourier transform of the uniform probability measure on the sphere S^{n-1}, $|\cdot|_2$ is the Euclidean norm in $\mathbb{R}^n$, and λ is a finite measure on $[0, \infty)$. In the same paper, Schoenberg proved an infinite dimensional version of this result: $f \in \Phi(\ell_2)$ if and only if

$$f(t) = \int_0^\infty \exp(-t^2 r^2)\ d\lambda(r).$$

Bretagnolle, Dacunha-Castelle and Krivine [BDK] proved a similar result for the classes $\Phi(\ell_q)$ for all $q \in (0, 2)$ (one just has to replace 2 by q in the formula), and showed that for $q > 2$ the classes $\Phi(\ell_q)$ (corresponding to infinite dimensional ℓ_q-spaces) are trivial, i.e. contain constant functions only. Cambanis, Keener and Simons [CKS] obtained a similar representation for the classes $\Phi(\ell_1^n)$. Richards [R] and Gneiting [G] partially characterized the classes $\Phi(\ell_q^n)$ for $0 < q < 2$. Aharoni, Maurey and Mityagin [AMM] proved that if E is an infinite dimensional Banach space with a symmetric basis $\{e_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \frac{\|e_1 + \cdots + e_n\|}{n^{1/2}} = 0,$$

then the class $\Phi(E)$ is trivial. Misiewicz [M2] proved that for $n \geq 3$ the classes $\Phi(\ell_\infty^n)$ are trivial, and Lisitsky [Li1] and Zastavnyi [Z1], [Z2] showed the same for the classes $\Phi(\ell_q^n)$, $q > 2$, $n \geq 3$. One can find more related results and references in [M3], [K3].

In all the results mentioned above the classes $\Phi(K)$ appear to be non-trivial only if K is the unit ball of a subspace of L_q with $0 < q \leq 2$. An old conjecture, explicitly formulated for the first time by Misiewicz [M1], is that the class $\Phi(K)$ can be non-trivial only in this case. A slightly weaker conjecture was formulated by Lisitsky [Li2]: if the class $\Phi(K)$ is non-trivial, then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. The concept of embedding in L_0 was introduced and studied in [KKYY], the original conjecture of Lisitsky was in terms of the representation (2):

Definition 1. *We say that a space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if there exist a finite Borel measure μ on the sphere S^{n-1} and a constant $C \in \mathbb{R}$ so that, for every $x \in \mathbb{R}^n$,*

$$\ln \|x\|_K = \int_{S^{n-1}} \ln |(x, \xi)| \ d\mu(\xi) + C. \tag{2}$$

It is quite easy to confirm the conjectures of Misiewicz and Lisitsky under additional assumptions that f or its Fourier transform have finite moments of certain orders; see [Mi1], [Ku], [Li2], [K4].

In this article we prove the conjecture of Lisitsky in its full strength:

Theorem 1. *Let K be an origin symmetric star body in $\mathbb{R}^n$, $n \geq 2$ and suppose that there exists an even non-constant continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(\|\cdot\|_K)$ is a positive definite function on $\mathbb{R}^n$. Then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.*

Corollary 1. *If a function γ is the standard of an n-dimensional version of a random variable, then there exists an origin symmetric star body K in $\mathbb{R}^n$ such that $\gamma = \|\cdot\|_K$ and the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.*

In the last section of the paper we use known results about embedding in L_0 to point out rather general classes of normed spaces for which the classes Φ are trivial and whose norms cannot serve as the standard of an n-dimensional version.

2. Proof of Theorem 1

As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$ (Schwartz test functions), and by

$\mathcal{S}'(\mathbb{R}^n)$ the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ is a locally integrable function with power growth at infinity, then the action of f on ϕ is defined by

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)\ dx.$$

We say that a distribution is positive (negative) outside of the origin in $\mathbb{R}^n$ if it assumes non-negative (non-positive) values on non-negative test functions with compact support outside of the origin.

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function ϕ. A distribution is positive definite if its Fourier transform is a positive distribution.

We use the following Fourier analytic characterization of embedding in L_0 proved in [KKYY, Th.3.1]:

Proposition 1. *Let K be an origin symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if the Fourier transform of $\ln \|x\|_K$ is a negative distribution outside of the origin in $\mathbb{R}^n$.*

Now we are ready to start the proof of Theorem 1.

Proof of Theorem 1. We write $\|\cdot\|$ instead of $\|\cdot\|_K$. By Bochner's theorem, the function $f(\|\cdot\|)$ is the Fourier transform of a finite measure μ on $\mathbb{R}^n$. We can assume that $f(0) = 1$, and, correspondingly, μ is a probability measure. The function f is positive definite on $\mathbb{R}$, as the restriction of a positive definite function, therefore, $|f(t)| \le f(0) = 1$ for every $t \in \mathbb{R}$ (see [VTC, p.188]).

Let ϕ be an even non-negative test function supported outside of the origin in $\mathbb{R}^n$. For every fixed $t > 0$, the function $f(t\|\cdot\|)$ is positive definite on $\mathbb{R}^n$, so

$$\int_{\mathbb{R}} f(t\|x\|)\hat{\phi}(x)\ dx = \left\langle \left(f(t\|\cdot\|)\right)^{\wedge}, \phi(x) \right\rangle \ge 0. \tag{3}$$

For any $\varepsilon \in (0,1)$, the integral

$$g(\varepsilon) = \int_{\mathbb{R}^n} \left(\int_0^1 t^{-1+\varepsilon} f(t\|x\|)dt + \int_1^\infty t^{-1-\varepsilon} f(t\|x\|)dt \right) \hat{\phi}(x)dx \tag{4}$$

converges absolutely, because f is bounded by 1 and the function in parentheses is bounded by $2/\varepsilon$. By the Fubini theorem,

$$g(\varepsilon) = \int_0^1 t^{-1+\varepsilon} \left(\int_{\mathbb{R}^n} f(t\|x\|)\hat{\phi}(x)dx \right) dt$$

$$+\int_1^\infty t^{-1-\varepsilon}\left(\int_{\mathbb{R}^n} f(t\|x\|)\hat{\phi}(x)dx\right)dt,$$

so by (3) the function g is non-negative:

$$g(\varepsilon)\geq 0 \quad \text{for every} \quad \varepsilon\in(0,1). \tag{5}$$

Now we study the behavior of the function g, as $\varepsilon\to 0$. We have

$$g(\varepsilon)=\int_{\mathbb{R}^n}\left(\|x\|^{-\varepsilon}\int_0^{\|x\|} t^{-1+\varepsilon}f(t)\ dt+\|x\|^{\varepsilon}\int_{\|x\|}^\infty t^{-1-\varepsilon}f(t)\ dt\right)\hat{\phi}(x)dx$$

$$=\int_{\mathbb{R}^n}\frac{\|x\|^{-\varepsilon}-1}{\varepsilon}\varepsilon\left(\int_0^{\|x\|} t^{-1+\varepsilon}f(t)\ dt\right)\hat{\phi}(x)dx \tag{6}$$

$$+\int_{\mathbb{R}^n}\frac{\|x\|^{\varepsilon}-1}{\varepsilon}\varepsilon\left(\int_{\|x\|}^\infty t^{-1-\varepsilon}f(t)\ dt\right)\hat{\phi}(x)dx \tag{7}$$

$$+\int_{\mathbb{R}^n}\left(\int_0^{\|x\|} t^{-1+\varepsilon}f(t)\ dt+\int_{\|x\|}^\infty t^{-1-\varepsilon}f(t)\ dt\right)\hat{\phi}(x)dx. \tag{8}$$

We write

$$g(\varepsilon)=u(\varepsilon)+v(\varepsilon)+w(\varepsilon),$$

where u,v,w are the functions defined by (6), (7) and (8), respectively.

We start with the function w.

Lemma 1.

$$\lim_{\varepsilon\to 0} w(\varepsilon)=0.$$

Proof : We can assume that $\varepsilon<1/2$. Fix $a>0$. Since ϕ is supported outside of the origin, we have $\int_{\mathbb{R}^n}\hat{\phi}(x)dx=0$ and

$$\int_{\mathbb{R}^n}\left(\int_0^a t^{-1+\varepsilon}f(t)\ dt+\int_a^\infty t^{-1-\varepsilon}f(t)\ dt\right)\hat{\phi}(x)dx=0,$$

because the expression in parentheses is a constant. Subtracting this from (8) we get

$$w(\varepsilon)=\int_{\mathbb{R}^n}\left(\int_a^{\|x\|}\left(t^{-1+\varepsilon}-t^{-1-\varepsilon}\right)f(t)\ dt\right)\hat{\phi}(x)dx.$$

Now for some $\theta(t,\varepsilon)\in[0,2\varepsilon]$,

$$t^{-1-\varepsilon}|t^{2\varepsilon}-1|=2\varepsilon\ t^{-1-\varepsilon}t^{\theta(t,\varepsilon)}|\ln t|$$

$$\leq 2\varepsilon(1+a^{-3/2}+\|x\|^{-3/2})(|\ln a|+\big|\ln\|x\|\big|),$$

so

$$|w(\varepsilon)| \leq 2\varepsilon \int_{\mathbb{R}^n} |\|x\| - a| \ (1 + a^{-3/2} + \|x\|^{-3/2})(|\ln a| + \big|\ln \|x\|\big|)|\hat{\phi}(x)|dx. \tag{9}$$

By the definition of a star body, K is bounded and contains a Euclidean ball with center at the origin, so there exist constants $c, d > 0$ so that for every $x \in \mathbb{R}^n$

$$c|x|_2 \leq \|x\| \leq d|x|_2, \tag{10}$$

where $|\cdot|_2$ is the Euclidean norm in $\mathbb{R}^n$. Note that $n \geq 2$ so $|\cdot|_2^{-3/2}$ is a locally integrable function on $\mathbb{R}^n$, $n \geq 2$. Also $\hat{\phi}$ is a test function and decreases at infinity faster than any power of the Euclidean norm. These facts, in conjunction with (10), imply that the integral in the right-hand side of (9) converges, which proves the lemma. □

We need the following elementary and well known fact.

Lemma 2. *Let h be a bounded integrable continuous at 0 function on $[0, A]$, $A > 0$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^A t^{-1+\varepsilon} h(t) dt = h(0).$$

Proof : We can assume that $\varepsilon < 1$. We have

$$\varepsilon \int_0^A t^{-1+\varepsilon} h(t) dt$$

$$= \varepsilon \int_0^\varepsilon t^{-1+\varepsilon}(h(t) - h(0))dt + \varepsilon h(0) \int_0^\varepsilon t^{-1+\varepsilon} dt + \varepsilon \int_\varepsilon^A t^{-1+\varepsilon} h(t) dt.$$

The first summand is less or equal to

$$\varepsilon^\varepsilon \max_{t \in [0,\varepsilon]} |h(t) - h(0)| \to 0, \quad \text{as} \quad \varepsilon \to 0,$$

because h is continuous at 0. The second summand is equal to

$$h(0)\varepsilon^\varepsilon \to h(0), \quad \text{as} \quad \varepsilon \to 0.$$

The third summand is less or equal to

$$|A^\varepsilon - \varepsilon^\varepsilon| \max_{t \in [0,A]} |h(t)| \to 0, \quad \text{as} \quad \varepsilon \to 0. \quad \square$$

Now we compute the limit at infinity of the function

$$u(\varepsilon) = \int_{\mathbb{R}^n} \frac{\|x\|^{-\varepsilon} - 1}{\varepsilon} \varepsilon \left(\int_0^{\|x\|} t^{-1+\varepsilon} f(t)\ dt \right) \hat{\phi}(x) dx.$$

Lemma 3.

$$\lim_{\varepsilon \to 0}\ u(\varepsilon) = -f(0) \int_{\mathbb{R}^n} \ln \|x\| \hat{\phi}(x) dx.$$

Proof : Using the estimates

$$\Big| \frac{\|x\|^{-\varepsilon} - 1}{\varepsilon} \Big| = \Big| \frac{1}{\varepsilon} \int_0^{\varepsilon} \|x\|^{-\theta} \ln \|x\| d\theta \Big| \leq \big| \ln \|x\| \big| (1 + \|x\|^{-1})$$

and

$$\Big| \varepsilon \int_0^{\|x\|} t^{-1+\varepsilon} f(t) dt \Big| \leq \|x\|^{\varepsilon} \leq \|x\| + 1,$$

we see that the functions under the integral over $\mathbb{R}^n$ in $u(\varepsilon)$ are dominated by an integrable function

$$\big| \ln \|x\| \big| (1 + \|x\|^{-1})(\|x\| + 1) |\hat{\phi}(x)|$$

of the variable x on $\mathbb{R}^n$. Clearly, for $x \neq 0$,

$$\lim_{\varepsilon \to 0} \frac{\|x\|^{-\varepsilon} - 1}{\varepsilon} = -\ln \|x\|.$$

Also, by Lemma 2, for every $x \in \mathbb{R}^n,\ x \neq 0$

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^{\|x\|} t^{-1+\varepsilon} f(t) dt = f(0) = 1,$$

so the functions under the integral by x in $u(\varepsilon)$ converge pointwise to $-\ln \|x\| \hat{\phi}(x)$. The result follows from the dominated convergence theorem. □

Now recall that

$$v(\varepsilon) = \int_{\mathbb{R}^n} \frac{\|x\|^{\varepsilon} - 1}{\varepsilon} \varepsilon \left(\int_{\|x\|}^{\infty} t^{-1-\varepsilon} f(t)\ dt \right) \hat{\phi}(x) dx.$$

We have

$$\varepsilon \int_{\|x\|}^{\infty} t^{-1-\varepsilon} f(t)\ dt = \varepsilon \int_0^{1/\|x\|} t^{-1+\varepsilon} f(1/t) dt.$$

The difficulty is that we cannot apply Lemma 2 to compute the limit of the right-hand side of the latter equality, because the function $f(1/t)$ may be discontinuous at zero. However, we can avoid this difficulty as follows:

Lemma 4. *There exist a sequence $\varepsilon_k \to 0$ and a number $c < 1$ such that*

$$\lim_{k\to\infty} v(\varepsilon_k) = c\int_{\mathbb{R}^n} \ln\|x\|\hat{\phi}(x)\ dx.$$

Proof : By a dominated convergence argument, similar to the one used in the previous lemma, it is enough to prove that there exist a sequence $\varepsilon_k \to 0$ and a number $c < 1$ such that for every $x \in \mathbb{R}^n,\ x \neq 0$

$$\lim_{k\to\infty} \varepsilon_k \int_{\|x\|}^{\infty} t^{-1-\varepsilon_k} f(t)dt = c.$$

For every $x \neq 0$ we have

$$\left|\varepsilon\int_{1/\varepsilon}^{\|x\|} t^{-1-\varepsilon} f(t)\ dt\right| \leq \left|\|x\|^{-\varepsilon} - \varepsilon^{\varepsilon}\right| \to 0, \quad \text{as} \quad \varepsilon \to 0,$$

so it is enough to find a sequence ε_k and a number $c < 1$ such that

$$\lim_{k\to\infty} \psi(\varepsilon_k) = c < 1,$$

where

$$\psi(\varepsilon) = \varepsilon\int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt.$$

Since the function ψ is bounded by 1 on $(0,1)$, it suffices to prove that $\psi(\varepsilon)$ cannot converge to 1, as $\varepsilon \to 0$.

Suppose that, to the contrary, $\lim_{\varepsilon\to 0}\psi(\varepsilon) = 1$. We use the following result from [VTC, p. 205]: if μ is a probability measure on $\mathbb{R}^n$ and γ is the standard Gaussian measure on $\mathbb{R}^n$, then for every $t > 0$

$$\mu\{x \in \mathbb{R}^n :\ |x|_2 > 1/t\} \leq 3\int_{\mathbb{R}^n} (1 - \hat{\mu}(ty))\, d\gamma(y), \tag{11}$$

where $|\cdot|_2$ is the Euclidean norm on $\mathbb{R}^n$. Let μ be the measure satisfying $\hat{\mu} = f(\|\cdot\|)$. For every $\varepsilon \in (0,1)$, integrating (11) we get

$$\varepsilon\int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon}\mu\{x \in \mathbb{R}^n :\ |x|_2 > 1/t\}dt$$

$$\leq \int_{\mathbb{R}^n}\left(\varepsilon\int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon}(1 - f(t\|y\|))dt\right) d\gamma(y). \tag{12}$$

Now

$$\varepsilon\int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon}\mu\{x \in \mathbb{R}^n :\ |x|_2 > 1/t\}\ dt$$

$$= \varepsilon\int_{0}^{\varepsilon} t^{-1+\varepsilon}\mu\{x \in \mathbb{R}^n :\ |x|_2 > t\}\ dt.$$

and, by Lemma 2, the limit of the left-hand side of (12) as $\varepsilon \to 0$ is equal to $\mu(\mathbb{R}^n \setminus \{0\})$.

On the other hand, the functions

$$h_\varepsilon(y) = \varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon}(1 - f(t\|y\|))dt \tag{13}$$

are uniformly (with respect to ε) bounded by 2. Write these functions as

$$h_\varepsilon(y) = \varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon}(1 - f(t\|y\|))dt = \varepsilon^\varepsilon - \|y\|^\varepsilon \varepsilon \int_{\|y\|/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt$$

$$= \varepsilon^\varepsilon - (\|y\|^\varepsilon - 1)\varepsilon \int_{\|y\|/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt$$

$$-\varepsilon \int_{\|y\|/\varepsilon}^{1/\varepsilon} t^{-1-\varepsilon} f(t)dt - \varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt.$$

For every $y \neq 0$

$$\left|(\|y\|^\varepsilon - 1)\varepsilon \int_{\|y\|/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt\right| \leq \left|\|y\|^\varepsilon - 1\right| \left(\frac{\|y\|}{\varepsilon}\right)^{-\varepsilon} \to 0, \quad \text{as} \quad \varepsilon \to 0,$$

$$\left|\varepsilon \int_{\|y\|/\varepsilon}^{1/\varepsilon} t^{-1-\varepsilon} f(t)dt\right| \leq \left|\varepsilon^\varepsilon - (\|y\|/\varepsilon)^{-\varepsilon}\right| \to 0, \quad \text{as} \quad \varepsilon \to 0,$$

and by our assumption

$$\varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt = \psi(\varepsilon) \to 1, \quad \text{as} \quad \varepsilon \to 0.$$

Therefore, the functions h_ε converge to zero pointwise as $\varepsilon \to 0$ and are uniformly bounded by a constant. By the dominated convergence theorem, the limit of the right-hand side of (12) is equal to 0, as $\varepsilon \to 0$.

Sending $\varepsilon \to 0$ in (12), we get $\mu(\mathbb{R}^n \setminus \{0\}) = 0$, therefore the probability measure μ is a unit atom at the origin and f is a constant function, which contradicts to the assumption of Theorem 1. □

End of the proof of Theorem 1: Let ε_k be the sequence from Lemma 4. Recall that g is a non-negative function (see (5)). By Lemmas 1, 3, 4,

$$0 \leq \lim_{k \to \infty} g(\varepsilon_k) = \lim_{k \to \infty} (u + v + w)(\varepsilon_k) = (-1 + c) \int_{\mathbb{R}^n} \ln \|x\| \hat{\phi}(x)dx,$$

where $c < 1$. Therefore,

$$\left\langle (\ln \| \cdot \|)^\wedge, \phi \right\rangle = \int_{\mathbb{R}^n} \ln \|x\| \hat{\phi}(x)dx \leq 0$$

for every even non-negative test function ϕ supported outside of the origin. By Proposition 1, $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_0. $\square$

3. Examples

The concept of embedding of a normed space in L_0 was studied in [KKYY]. In particular, it was proved in [KKYY, Th.6.7] that

Proposition 2. *Every finite dimensional subspace of $L_p,\ 0 < p \le 2$ embeds in L_0.*

On the other hand, as proved in [KKYY,Th.6.3],

Proposition 3. *If $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0, it also embeds in L_p for every $-n < p < 0$.*

The definition and properties of embeddings in $L_p,\ p < 0$ and their connections with geometry can be found [K3, Ch. 6]. Propositions 2 and 3 confirm the place of L_0 in the scale of L_p-spaces. Speaking informally, the space L_0 is larger than every $L_p,\ p \in (0,2)$, but smaller than every $L_p,\ p < 0$.

There are many examples of normed spaces that embed in L_0, but don't embed in $L_p,\ p \in (0,2)$ (see [KKYY, Th. 6.5]). In particular, the spaces $\ell_q^3,\ q > 2$ have this property. In fact, every three dimensional normed space embeds in L_0 (see [KKYY, Corollary 4.3]). However, starting from dimension 4, there are many normed spaces that do not embed in L_0. The following result from [K3, Th. 4.19] essentially shows that a normed space with dimension greater than 4 does not embed in L_0 if the second derivative of its norm at zero in at least one direction is equal to 0.

Proposition 4. *Let $n \ge 4,\ -n < p < 0$ and let $X = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional normed space with a normalized basis $e_1, \dots, e_n$ so that:*
(i) For every fixed $(x_2, \dots, x_n) \in \mathbb{R}^{n-1} \setminus \{0\}$, the function

$$x_1 \mapsto \|x_1 e_1 + \sum_{i=2}^n x_i e_i\|$$

has a continuous second derivative everywhere on $\mathbb{R}$, and

$$\|x\|'_{x_1}(0, x_2, \dots, x_n) = \|x\|''_{x_1^2}(0, x_2, \dots, x_n) = 0,$$

where $\|x\|'_{x_1}$ and $\|x\|''_{x_1^2}$ stand for the first and second partial derivatives by x_1 of the norm $\|x_1 e_1 + \cdots + x_n e_n\|$.

(ii) There exists a constant C so that, for every $x_1 \in \mathbb{R}$ and every $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ with $\|x_2e_2 + \cdots + x_ne_n\| = 1$, one has

$$\|x\|''_{x_1^2}(x_1, x_2, \dots, x_n) \leq C.$$

(iii) Convergence in the limit

$$\lim_{x_1 \to 0} \|x\|''_{x_1^2}(x_1, x_2, \dots, x_n) = 0$$

is uniform with respect to $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ satisfying the condition $\|x_2e_2 + \cdots + x_ne_n\| = 1$.

Then the space $(\mathbb{R}^n, \|\cdot\|)$ does not embed in L_0.

Proof : It was proved in [K3, Th. 4.19] that under the assumptions of Proposition 4 the function $\|\cdot\|_K^{-p}$ represents a positive definite distribution if and only if $p \in (n-3, n]$. By [K3, Th. 6.15] the space $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in $L_p,\ p \in (-1, 0)$, so it also does not embed in L_0 by Proposition 3. The result follows from Theorem 1. $\square$

From Proposition 4 and Theorem 1 we immediately get

Corollary 2. *If a normed space $(\mathbb{R}^n, \|\cdot\|),\ n \geq 4$ satisfies the conditions of Proposition 4, then a function of the form $f(\|\cdot\|)$ can be positive definite only if f is a constant function. The norm of such a space cannot appear as the standard of an n-dimensional version.*

Let us give several examples of spaces satisfying the conditions of Proposition 4. For normed spaces X and Y and $q \in \mathbb{R},\ q \geq 1$, the q-sum $(X \oplus Y)_q$ of X and Y is defined as the space of pairs $\{(x, y) : x \in X, y \in Y\}$ with the norm

$$\|(x, y)\| = \left(\|x\|_X^q + \|y\|_Y^q\right)^{1/q}.$$

It was proved in [K2, Th 2] that such spaces with $q > 2$ satisfy the conditions of Proposition 4 provided that the dimension of X is greater or equal to 3.

Another example is that of Orlicz spaces. Recall that an *Orlicz function* M is a non-decreasing convex function on $[0, \infty)$ such that $M(0) = 0$ and $M(t) > 0$ for every $t > 0$. The norm $\|\cdot\|_M$ of the n-dimensional Orlicz space ℓ_M^n is defined implicitly by the equality

$$\sum_{k=1}^{n} M(|x_k|/\|x\|_M) = 1,\ x \in \mathbb{R}^n \setminus \{0\}.$$

As shown in [K2, Th 3], the spaces $\ell_M^n,\ n \geq 4$ satisfy the conditions of Proposition 4 if the Orlicz function $M \in C^2([0, \infty))$ is such that $M'(0) = M''(0) = 0$.

Corollary 3. *If a normed space* $(\mathbb{R}^n, \|\cdot\|)$ *contains a subspace isometric to* $(X \oplus Y)_q$*, where* $q > 2$ *and the dimension of* X *is at least 3, or contains an Orlicz space* ℓ^4_M*, where* M *is an Orlicz function such that* $M \in C^2([0,\infty))$ *and* $M'(0) = M''(0) = 0$*, then a function of the form* $f(\|\cdot\|)$ *can be positive definite only if* f *is a constant function.*

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